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ASYMPTOTIC ANALYSIS OF BOUNDARY AND INITIAL CONDITIONS IN THE DYNAMICS OF THIN PLATES

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Asymptotic analysis of three-dimensional dynamic equations of a thin plate was used earlier in [1] to construct the inner state of stress and to indicate the ways of obtaining more accurate results. The present paper investigates the problem of obtaining, by means of asymptotic methods, valid boundary and initial conditions corresponding to the two-dimensional dynamic equations, and of establishing their asymptotic accuracy.

1. Consider the problem of reducing the three-dimensional dynamic problem of the theory of elasticity for a thin plate, to the two-dimensional one, with the problem of satisfying the boundary and initial conditions included. We note that in considering the inner problem in [1], the author used the initial ratio of the displacement intensities corresponding to the bending motions of the plate. For this reason, the deformation in the plane of the plate was of secondary importance. If the plate is homogeneous, then the bending is completely separate from the deformation in the plane of the plate in both static and dynamic cases, and their relative intensities are in no way dependent on each other.

We also note that in reducing the three-dimensional problem to the two-dimensional one, there is no need to define *a priori* the asymptotics of the state of stress in question so as to confirm subsequently that the initial assumptions were correct. We cannot deal with this problem in more detail, but we can show that either one, or the other asymptotics follows automatically from the assumption made about the way in which the state of stress in question varies in different directions.

Let us consider, in a norrow region of length $2l$ and thickness $2h$, a phenomenon with characteristic dimension of the deformation patterns l_0, and characteristic time t_0. Obviously we can expect any significant simplifications in the initial problem of the theory of elasticity only in the case of phenomena for which l_0 is much greater than h, and the time t_0 is much longer than $h\sqrt{\rho / E}$, the latter commensurable with the time in which the perturbation traverses the distance h in an elastic medium. An inner state of stress satisfies these conditions, and its determination can be reduced to an iterative process at each stage of which certain two-dimensional equations must be solved. In constructing this state of stress, we take into account the equations of the theory of elasticity and the boundary conditions in terms of the stresses at the face planes. In order to satisfy the boundary conditions at the side surface and the initial conditions of the original problem of the theory of elasticity and to formulate the boundary and initial conditions of the inner problem, we introduce certain, rapidly changing states of stress which, in a certain sense, exert little influence on the inner state of stress.

2. Let us introduce the dimensionless variables. We refer the displacements to h, and the stresses to the modulus of elasticity E. We investigate the states of stress varying in different manner in different directions, remembering that our arguments must be relevant to the inner problem, as well as to the rapidly changing states of stress. Let the quantities r_1, r_2 and r_3 $(r_1 = p_1 / q, r_2 = p_2 / q$ where p_1, p_2 and q are integers) characterize the variability of the state of stress over the space variables x, y and z. This means that for each variable there is a corresponding characteristic dimension $l_i = l\varepsilon^{r_i}$ $(i = 1, 2, 3)$. Since the characteristic dimension in the direction of the z-axis is h, we have $r_3 = 1$. The characteristic time t_0 is given by

$$t_0 = \varepsilon^{\omega-1} l \sqrt{\rho / E}$$

and the parameter ω characterizes the variability of the state of stress with time.

Let us carry out the scale expansion transformation, referring the variables x, y, z and t to the corresponding characteristic dimensions

$$\xi = \frac{x}{l_1}, \quad \eta = \frac{y}{l_2}, \quad \zeta = \frac{z}{h}, \quad \tau = \frac{t}{t_0}$$

Having passed to the new variables, we seek solutions of the equations obtained from the Lamé equations, for any states of stress, in the form of asymptotic series in terms of the small parameter $\lambda = \varepsilon^{1/q}$

$$v_x = \varepsilon^{\varkappa} \sum_{s=0} \lambda^s v_x^{(s)}, \ldots \tag{2.1}$$

and using the relations of Hooke's Law we obtain similar expressions for the stresses.

The choice of the quantity $\varkappa$ in (2.1) is somewhat conditional. For example, in considering the inner state of stress it can be related to the intensity of the external forces. The main problem which must not be disregarded consists of the fact that, in the expansions of the type (2.1) for the stresses and displacements, some of the first terms vanish identically. The number of such terms differs for the different stress and displacement components, and is automatically defined by the character of the state of stress in question. We can state it more accurately by saying that the number depends on the variability of the state of stress in different directions. Determination of the number of the first, identically vanishing terms in the expansions of the type (2.1) will lead, in the end, to establishing the asymptotics of the state of stress in question, since the intensities of all the stresses and displacements will then become known.

3. The inner state of stress changes little in the plane of the plate, and we have $0 \leqslant r_i < 1$. We shall restrict ourselves, for simplicity, to a single variability index r in the plane of the plate, using the larger of r_1 and r_2 as its value. The choice of the value of $\varkappa$ in (2.1) can be made dependent on the condition that a normal load applied to the face planes is independent of the relative thickness of the plate. Then, for the deformation in the plane of the plate, we have $\varkappa = -3 + 3r$

and for the bending we have $\varkappa = -4 + 4r$.

We can easily obtain the relation connecting the parameter ω characterizing the variability of the process with time with the value of r, by making certain that the inertial terms appear in the two-dimensional equations of the inner state of stress in the zeroth approximation. We have $\omega = 1 + r$ for the deformation in the plane of the plate, and $\omega = 2r$ for the bending.

4. A state of stress in the boundary layer which appears near the plate edge [2, 3], diminishes rapidly with increasing distance from the edge. Introduction of this state of stress makes it possible to satisfy the boundary conditions, formulated in terms of the theory of elasticity, at the side surface of the plate, to establish the boundary conditions of the inner problem, and to determine more accurately the state of stress near the edge.

The boundary layer near the edge $x = 0$ shows a large degree of variability in the x- and z-directions, and same variability as the inner state of stress in the y-direction and with time. Thus we have the following values for the variability indices in the x, y, z and t directions:

$$r_1 = 1, \quad r_2 < 1, \; r_3 = 1, \quad 0 \leqslant \omega < 2$$

The value of $\varkappa$ in (2.1) will be chosen equal to the highest order of the quantities of the inner problem appearing in the boundary conditions at the side surface. We obtain $v_x^{(s)}$ and $v_z^{(s)}$ from the following equations of plane deformation for a half-strip:

$$(\lambda^* + \mu^*)\frac{\partial}{\partial \xi}\left(\frac{\partial v_x^{(s)}}{\partial \xi} + \frac{\partial v_z^{(s)}}{\partial \zeta}\right) + \mu^* \Delta v_x^{(s)} = \frac{\partial^2 v_x^{(s-4q+2\omega q)}}{\partial \tau^2} - R_\xi^{(s)}$$

$$(\lambda^* + \mu^*)\frac{\partial}{\partial \zeta}\left(\frac{\partial v_x^{(s)}}{\partial \xi} + \frac{\partial v_z^{(s)}}{\partial \zeta}\right) + \mu^* \Delta v_z^{(s)} = \frac{\partial v_z^{(s-4q+2\omega q)}}{\partial \tau^2} - R_\zeta^{(s)}$$

$$R_\xi^{(s)} = (\lambda^* + \mu^*)\frac{\partial^2 v_y^{(s-q+p)}}{\partial \xi\, \partial \eta} + \mu^* \frac{\partial^2 v_x^{(s-2q+2p)}}{\partial \eta^2}$$

$$R_\zeta^{(s)} = (\lambda^* + \mu^*)\frac{\partial^2 v_y^{(s-q+p)}}{\partial \eta \partial \zeta} + \mu^* \frac{\partial^2 v_x^{(s-2q+2p)}}{\partial \eta^2}$$

Here λ^* and μ^* are the Lamé constants referred to the modulus of elasticity E. They can be expressed in terms of Poisson's ratio ν in the form

$$\lambda^* = \frac{\nu}{(1+\nu)(1-2\nu)}, \quad \mu^* = \frac{1}{2(1+\nu)}$$

The following conditions hold on the face planes:

$$\left[\frac{\partial v_x^{(s)}}{\partial \zeta} + \frac{\partial v_z^{(s)}}{\partial \xi}\right]_{\zeta=\pm 1} = 0$$

$$\left[\frac{\partial v_z^{(s)}}{\partial \zeta} + \frac{\nu}{1-\nu}\frac{\partial v_x^{(s)}}{\partial \xi} + \frac{\nu}{1-\nu}\frac{\partial v_y^{(s-q-p)}}{\partial \eta}\right]_{\zeta=\pm 1} = 0$$

and the determination of $v_y^{(s)}$ is reduced to solving the antiplane problem for a half-strip

$$\mu^* \left(\frac{\partial^2 v_y^{(s)}}{\partial \xi^2} + \frac{\partial^2 v_y^{(s)}}{\partial \zeta^2}\right) = \frac{\partial^2 v_y^{(s-4q+2\omega q)}}{\partial \tau^2} - R_\eta^{(s)}$$

$$R_\eta^{(s)} = (\lambda^* + \mu^*)\left[\frac{\partial}{\partial \eta}\left(\frac{\partial v_x^{(s-q+p)}}{\partial \xi} + \frac{\partial v_z^{(s-q+p)}}{\partial \zeta}\right) + 2(1-\nu)\frac{\partial^2 v_y^{(s-2q+2p)}}{\partial \eta^2}\right]$$

$$\left[\frac{\partial v_y^{(s)}}{\partial \zeta} + \frac{\partial v_z^{(s-q+p)}}{\partial \eta}\right]_{\zeta=\pm 1} = 0$$

Thus the determination of the boundary layer in the dynamic as well as the static problem separates into solutions of the plane and antiplane problems for a half-strip. For the dynamic processes for which $0 \leqslant \omega < 2$, both these problems have a quasi-static character. The inertial terms do not in general appear in the equations of a series of first approximations, and in the approximations in which they do appear, they are determined in terms of the quantitites obtained from the preceding approximations.

It can be shown that the two-dimensional dynamic equations of the inner problem corresponding to the accuracy of $O(\varepsilon^{2-2r})$ have the same boundary conditions as those of the static case [2]. It follows that for the dynamic problems not only the boundary conditions of the classical theory, but also the stronger conditions [2] have the same form as those of the static problems. The boundary conditions corresponding to the classical theory are of accuracy of $O(\varepsilon^{1-r})$, and the stronger boundary conditions of $O(\varepsilon^{2-2r})$.

5. Next we consider the problem of introducing a rapidly changing state of stress which, although it does not, in a certain sense, exert any appreciable influence on the inner state of stress, nevertheless it allows both states to satisfy the initial conditions of the initial problem of the theory of elasticity.

In the four-dimensional space x, y, z, t the plane $t = 0$ represents, for the phenomenon under consideration, a kind of a boundary. This suggests a possibility of introducing into our discussion an auxilliary state of stress with large variability with respect to time, but with the same variability in the directions of the x, y -axes as the inner state of stress. Thus the variability indices for the auxilliary state of stress in the directions of the x, y, z and t axes are $r_1 < 1$, $r_2 < 1$, $r_3 = 1$, $\omega = 2$.

We choose the value of $\varkappa$ in (2.1) to be equal to the largest order of displacements in the inner problem, and we obtain the following equations for $v_x^{(s)}$ (xy) and $v_z^{(s)}$:

$$\frac{\partial^2 v_x^{(s)}}{\partial \zeta^2} - 2(1+\nu)\frac{\partial^2 v_x^{(s)}}{\partial \tau^2} = -R_\xi^{(s)} \quad (xy)$$

$$\frac{\partial^2 v_z^{(s)}}{\partial \zeta^2} - \frac{(1+\nu)(1-2\nu)}{1-\nu}\frac{\partial^2 v_z^{(s)}}{\partial \tau^2} = -R_\zeta^{(s)}$$

$$R_\xi^{(s)} = \frac{1}{2(1-\nu)}\frac{\partial^2 v_z^{(s-q+p)}}{\partial\xi\partial\zeta} + \frac{1}{2(1-\nu)} \times$$
$$\frac{\partial}{\partial\xi}\left(\frac{\partial v_x^{(s-2q+2p)}}{\partial\xi} + \frac{\partial v_y^{(s-2q+2p)}}{\partial\eta}\right) + \Delta v_x^{(s-2q+2p)} \, (xy)$$

$$R_\zeta^{(s)} = \frac{1}{2(1-\nu)}\frac{\partial}{\partial\zeta}\left(\frac{\partial v_x^{(s-q+p)}}{\partial\xi} + \frac{\partial v_y^{(s-q+p)}}{\partial\eta}\right) + \frac{1-2\nu}{2(1-\nu)}\Delta v_z^{(s-2q+2p)}$$

When $\zeta = \pm 1$, zero stress conditions must prevail, i. e.

$$\left.\frac{\partial v_x^{(s)}}{\partial\zeta}\right|_{\zeta=\pm1} + \left.\frac{\partial v_z^{(s-q+p)}}{\partial\xi}\right|_{\zeta=\pm1} = 0 \quad (xy)$$
$$\left.\frac{\partial v_z^{(s)}}{\partial\zeta}\right|_{\zeta=\pm1} + \frac{\nu}{1-\nu}\left.\left(\frac{\partial v_x^{(s-q+p)}}{\partial\xi} + \frac{\partial v_y^{(s-q+p)}}{\partial\eta}\right)\right|_{\zeta=\pm1} = 0$$

Thus the process of constructing an auxilliary state of stress in any approximation is reduced to that of solving the wave equations under certain initial and boundary conditions. The equations and boundary conditions are homogeneous when $s < q - p$ and become, in general, inhomogeneous when $s > q - p$.

We note that in the case of the state of stress under consideration the propagation of perturbations for the transverse displacements (relative to the plate) occurs with the velocity of longitudinal waves, and for the displacements in the plane of the plate, with the velocity of transverse waves.

We shall investigate the bending and deformation in the plane of the plate separately. In the bending case $v_x^{(s)}$ and $v_y^{(s)}$ are odd functions and $v_z^{(s)}$ is an even function of ζ. In the case of deformation in the plane of the plate the relationship is reversed, $v_x^{(s)}$ and $v_y^{(s)}$ being even and $v_z^{(s)}$ odd. Therefore, to construct an auxilliary state of stress it is sufficient to consider the wave equations for a function even and odd in ζ. The problem has already been studied in detail, and its results should be used in formulating the initial conditions of the two-dimensional problems.

6. We have reduced the construction of the auxilliary state of stress to solving the wave equation (6.1) with the boundary and initial conditions (6.2) and (6.3)

$$\frac{\partial^2 u}{\partial\zeta^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial\tau^2} = \varphi(\zeta, \tau) \tag{6.1}$$
$$\left.\frac{\partial u}{\partial\zeta}\right|_{\zeta=\pm1} = \psi(\tau) \tag{6.2}$$
$$u(\zeta, 0) = f(\zeta), \quad \left.\frac{\partial u}{\partial\tau}\right|_{\tau=0} = F(\zeta) \tag{6.3}$$

The solution of this problem consists of a solution of the homogeneous equation satisfying the null boundary condition and specified initial conditions, and of the solution of the inhomogeneous equation with inhomogeneous boundary conditions and zero initial conditions.

Let $u(\zeta, \tau)$ be an even function of ζ. If $\varphi(\zeta, \tau) = 0$, $\psi(\tau) = 0$, then the solution of the wave equation has the form

$$u(\zeta, \tau) = u_0 + u_1\tau + u^*(\zeta, \tau) \tag{6.4}$$
$$u_0 = \int_0^1 f(\zeta)\,d\zeta, \quad u_1 = \int_0^1 F(\zeta)\,d\zeta$$

The first term of (6.4) yields a part of the solution independent of time, the second term a part linearly dependent on time, and the third term a part oscillatory with respect to time. The solution $u(\zeta, \tau)$ will be pure oscillatory with respect to time if and only if the mean values of the initial conditions (6.3) vanish, i.e. if

$$\int_0^1 f(\zeta)\,d\zeta = 0, \quad \int_0^1 F(\zeta)\,d\zeta = 0 \tag{6.5}$$

and we shall call (6.5) for brevity the oscillatory conditions.

If $\varphi(\zeta, \tau)$, $\psi(\tau)$ in (6.1) and (6.2) are not zero but purely oscillatory functions with respect to time, then the solution of (6.1) with conditions (6.2) and zero initial conditions is obviously a function purely oscillatory with respect to time.

Let $u(\zeta, \tau)$ be an odd function of ζ. Then the solution of (6.1) with conditions (6.2) and (6.3) will always be a function purely oscillatory with respect to time. Thus in the case of the functions odd in ζ, the solution of the wave equation will always be oscillatory with respect to time. For the even functions of ζ this will be true only when the conditions of oscillation are met, i.e. when the mean values of the initial conditions vanish.

7. Le us now consider the problem of satisfying the initial conditions of the parent problem of the theory of elasticity, of formulating the initial conditions of the two-dimensional inner problem, and of establishing their asymptotic accuracy, using the example of transverse motions of the plate, the motions caused by a discontinuous change in the surface loading at the instant $t = 0$.

Let the plate be at rest when $t < 0$, but be at the same time deformed by the action of some surface load. When the surface load changes discontinuously at the instant $t = 0$, the plate is set in motion. In the case of the initial problem of the theory of elasticity, the three displacement components assume at the initial instant their prescribed values, and the three velocity components vanish. The following boundary conditions hold at the face planes of the plate:

$$E\sigma_{xz}|_{\zeta=\pm 1} = \tau_x^-\ (xy), \quad E\sigma_z|_{\zeta=\pm 1} = \pm\tau_z^-\quad (t < 0)$$
$$E\sigma_{xz}|_{\zeta=\pm 1} = \tau_x^+\ (xy), \quad E\sigma_z|_{\zeta=\pm 1} = \pm\tau_z^+\quad (t > 0)$$

The inner state of stress of the plate can be represented, for both $t < 0$ and $t > 0$, in the form of asymptotic series of the form (2.1) in which $\varkappa = -4q + 4p$. The quantities $v_x^{(s)}(xy)$ and $v_z^{(s)}$ are polynomials in ζ, and their degree increases with increasing order of approximation. We have

$$
\begin{aligned}
&v_x^{(0)} = 0 && (xy), \quad v_z^{(0)} = v_{z0}^{(0)} \\
&v_x^{(q-p)} = \zeta v_{x1}^{(q-p)} && (xy), \quad v_z^{(q-p)} = v_{z0}^{(q-p)} \\
&v_x^{(2q-2p)} = \zeta v_{x1}^{(2q-2p)} && (xy), \quad v_z^{(2q-2p)} = v_{z0}^{(2q-2p)} + \zeta^2 v_{z2}^{(2q-2p)} \\
&v_x^{(3q-3p)} = \zeta v_{x1}^{(3q-3p)} + \zeta^3 v_{x3}^{(3q-3p)} && (xy), \quad v_z^{(3q-3p)} = v_{z0}^{(3q-3p)} + \zeta^2 v_{z2}^{(3q-3p)} \\
&\ldots && v_z^{(4q-4p)} = v_{z0}^{(4q-4p)} + \zeta^2 v_{z2}^{(4q+4p)} + \zeta^4 v_{z4}^{(4q-4p)}
\end{aligned}
\tag{7.1}
$$

etc. At $\tau = 0$ all quantities except $v_{x1}^{(3q-3p)}\,(xy)$, $v_{z2}^{(4q-4p)}$ appearing in the right hand sides of (7.1) are continuous. The excepted quantities undergo a jump at $\tau = 0$, the magnitude of which is given by the jump in the value of the surface load

$$
v_{x1}^{(3q-3p)}\Big|_{\tau=-0} - v_{x1}^{(3q-3p)}\Big|_{\tau=+0} = 2(1+\nu)\,\lambda^{q-p}\,\frac{\tau_x^- - \tau_x^+}{E} \qquad (xy)
$$

$$
v_{z2}^{(4q-4p)}\Big|_{\tau=-0} - v_{z2}^{(4q-4p)}\Big|_{\tau=+0} = \frac{1+\nu}{1-\nu}\,\frac{\lambda^{q-p}}{E} \times \left(\frac{\partial(\tau_x^- - \tau_x^+)}{\partial\xi} + \frac{\partial(\tau_y^- - \tau_y^+)}{\partial\eta}\right)
$$

The appearance of the factor λ^{q-p} in the above conditions indicates that the contribution of the tangential surface forces can be compared with the contribution of the normal surface forces only in the case when $\lambda^{q-p}\tau_x \sim \tau_z$. Since by definition $\tau_z \sim \lambda^\circ$, we have $\tau_x \sim \lambda^{-q+p}$.

Let us now bring in the problems $A_x^{(s)}\,(xy)$ and $A_z^{(s)}$ referring, respectively, to the determination of the displacements $V_x^{(s)}\,(xy)$ and $V_z^{(s)}$ for the state of stress purely oscillatory with respect to time.

It can be shown that for $s < 3q - 3p$ the solutions of the problems $A_x^{(s)}\,(xy)$ vanish identically since they reduce to the problem of solving homogeneous equations with homogeneous initial and boundary conditions. The solution $A_x^{(3q-3p)}$ on the other hand is not zero, since it satisfies the homogeneous equations and boundary conditions with nonzero initial conditions.

The solutions of the problems $A_z^{(s)}$ satisfy, for $s < 4q - 4p$ the homogeneous equations, homogeneous boundary conditions and the following initial conditions:

$$
V_z^{(s)}\Big|_{\tau=0} = v_{z0}^{(s)}\Big|_{\tau=+0} - v_{z0}^{(s)}\Big|_{\tau=-0}, \qquad \frac{\partial V_z^{(s)}}{\partial\tau}\Big|_{\tau=0} = 0
$$

The requirement that the solutions of the problems $A_z^{(s)}$ $(s < 4q - 4p)$ satisfy the conditions of oscillation (6.5), yields the initial conditions of the two-dimensional inner problem for $\tau > 0$

$$
v_{z0}^{(s)}\Big|_{\tau=+0} = v_{z0}^{(s)}\Big|_{\tau=-0}, \qquad \frac{\partial v_{z0}^{(s)}}{\partial\tau}\Big|_{\tau=+0} = 0 \tag{7.2}
$$

After satisfying the conditions of oscillation, the initial conditions of the problems $A_z^{(s)}$ $(s < 4q - 4p)$ become homogeneous and the solutions of these problems vanish identically.

Let us consider the problem $A_z^{(4q-4p)}$ which reduces to that of solving a homogeneous wave equation with homogeneous boundary conditions and the following initial conditions:

$$V_z^{(4q-4p)}\Big|_{\tau=0} = v_{z0}^{(4q-4p)}\Big|_{\tau=-0} - v_{z0}^{(4q-4p)}\Big|_{\tau=+0} + \tag{7.3}$$

$$\zeta^2 \frac{1+\nu}{1-\nu}\frac{\lambda^{q-p}}{E}\left[\frac{\partial(\tau_x^- - \tau_x^+)}{\partial\xi} + \frac{\partial(\tau_y^- - \tau_y^+)}{\partial\eta}\right], \quad \frac{\partial V_z^{(4q-4p)}}{\partial\tau}\Big|_{\tau=0} = 0$$

The condition that the solution sought obeys the conditions of oscillation (6.5), yields the stronger initial conditions for the two-dimensional inner problem

$$v_{z0}^{(4q-4p)}\Big|_{\tau=+0} = v_{z0}^{(4q-4p)}\Big|_{\tau=-0} + \frac{1}{3}\frac{1+\nu}{1-\nu}\frac{\lambda^{q-p}}{E} \times \tag{7.4}$$

$$\left[\frac{\partial(\tau_x^- - \tau_x^+)}{\partial\xi} + \frac{\partial(\tau_y^- - \tau_y^+)}{\partial\eta}\right], \quad \frac{\partial v_{z0}^{(4q-4p)}}{\partial\tau}\Big|_{\tau=+0} = 0$$

The initial conditions (7.3) of the problem $A_z^{(4q-4p)}$ with (7.4) taken into account, assume the form

$$V_z^{(4q-4p)}\Big|_{\tau=0} = (1-3\zeta^2)\left(v_{z0}^{(4q-4p)}\Big|_{\tau=-0} - v_{z0}^{(4q-4p)}\Big|_{\tau=+0}\right)$$

From the initial conditions (7.2), (7.4) specified for different approximations, we can pass to the initial conditions not containing any approximation indices, but determined to within the same asymptotic accuracy. The initial conditions corresponding to the two-dimensional equations of the inner problem have the form

$$u_{z0}\big|_{t=+0} = u_{z0}\big|_{t=-0}, \quad \frac{\partial u_{z0}}{\partial t}\Big|_{t=+0} = 0 \tag{7.5}$$

with the accuracy of $O(\varepsilon^{4-4r})$ and

$$u_{z0}\big|_{t=+0} = u_{z0}\big|_{t=-0} + h^2\frac{1}{3E}\frac{1+\nu}{1-\nu} \times \tag{7.6}$$

$$\left[\frac{\partial(\tau_x^- - \tau_x^+)}{\partial x} + \frac{\partial(\tau_y^- - \tau_y^+)}{\partial y}\right], \quad \frac{\partial u_{z0}}{\partial t}\Big|_{t=+0} = 0$$

with the accuracy greater than $O(\varepsilon^{4-4r})$. Here . $u_{z0} = hv_{z0}$.

Thus, using the asymptotic approach to solve the bending problem, we have obtained only two specified initial conditions irrespective of the degree of accuracy; we specify, at the initial instant of time, the transverse displacements and transverse velocities of the points of the middle plane.

The number of initial conditions (7.5) with asymptotic accuracy of $O(\varepsilon^{4-4r})$

is in full correspondence with the type of the two-dimensional equation of the inner problem corresponding to the same degree of asymptotic accuracy. For the higher degree of accuracy (higher than $O(\varepsilon^{4-4r})$) such a correspondence no longer exists. We have two sharpened initial conditions (7.6), but the two-dimensional equation of the inner problem contains a fourth order time derivative.

This confirms once again the need for using an iterative process to obtain more accurate results, in the investigations of thin plates behavior. When the problem is solved by iterative methods, there is a full correspondence between the type of the equation and the number of initial and boundary conditions at each stage.

We note that the stresses $\sigma_{xz}(xy)$, σ_{zz} are of the same order for the oscillatory state of stress, as for the inner problem. When the stresses $\sigma_{xx}(xy)$, σ_{xy} are determined with the accuracy exceeding that of the Kirchhoff—Love hypothesis, then the oscillatory state of stress must also be taken into account.

8. We shall pause briefly to formulate the prescribed initial conditions and to establish their asymptotic accuracy for the two-dimensional dynamic problem of deformation of a plate in its plane. As an example, we shall consider the problem of motion arising in the plane of the plate as a result of a jump in the value of the surface load. Let the boundary conditions at the face planes of the plate have the form

$$E\sigma_{xz}|_{\zeta=\pm 1} = \pm\tau_x^-\quad (xy),\quad E\sigma_z|_{\zeta=\pm 1} = \tau_z^-\quad (t<0)$$
$$E\sigma_{xz}|_{\zeta=\pm 1} = \pm\tau_x^+\quad (xy),\quad E\sigma_z|_{\zeta=\pm 1} = \tau_z^+\quad (t>0)$$

We use the same reasoning as in the case of the bending motions of the plate. For plane motions of the plate we find that the asymptotic approach yields, for each approximation, four initial conditions for two translations and two plane velocities of the plate, the number corresponding to the character of the two-dimensional dynamic equations. We have established that the initial conditions

$$u_{x0}|_{t=+0} = u_{x0}|_{t=-0}\quad (xy),\quad \left.\frac{\partial u_{x0}}{\partial x}\right|_{t=+0} = 0\quad (xy)$$

have asymptotic accuracy of $O(\varepsilon^{4-4r})$.

We have also obtained the stronger initial conditions with accuracy higher than $O(\varepsilon^{4-4r})$. For the problem in question these conditions have the form

$$u_{x0}|_{t=+0} = u_{x0}|_{t=-0} - \frac{1+\nu}{6(1-\nu)}\,\frac{1}{E}\,\frac{\partial(\tau_z^- - \tau_z^+)}{\partial x}\quad (xy),$$
$$\left.\frac{\partial u_{x0}}{\partial t}\right|_{t=0} = 0\quad (xy)$$

We note that the order of the stresses σ_z for the oscillatory state of stress is the same as for the inner problem. The oscillatory state of stress should be taken into account when determining the stresses $\sigma_x(xy)$ and $\sigma_{xz}(xy)$ with the accuracy exceeding $O(\varepsilon^{2-2r})$.

REFERENCES

1. G u s e i n - Z a d e, M. I. Asymptotic analysis of three-dimensional dynamic equations of a thin plate. PMM Vol. 38, No. 6, 1974.

2. K o l o s, A. V., Methods of refining the classical theory of bending and extension of plates. PMM Vol. 29, No. 4, 1965.

3. G o l ' d e n v e i z e r, A. L. Boundary layer and its interaction with the interior state of stress of an elastic thin shell. PMM Vol. 35, No. 6, 1969.

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